Synchronizationlike phenomena in coupled stochastic bistable systems

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A model of two coupled bistable systems driven by independent noise sources is considered. The cases of mutual coupling as well as one-directional coupling are investigated. We find that in such stochastic systems effects similar to synchronization phenomena in classical oscillating systems can be observed. It is shown that when the strength of coupling achieves some critical value then the stochastic processes in the subsystems become coherent. The appearance of coherence corresponds to the bifurcation in the two-dimensional stationary probability density. Moreover, the effect of coincidence of the Kramers frequencies in the subsystems can be observed. The latter is similar to the synchronization via frequency locking in classical oscillating systems.

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Investigations in the field of the noise influence on the nonlinear dynamical systems demonstrate a number of nontrivial effects [1]. First of all, there are noise-induced transitions [2] and the phenomenon of stochastic resonance [3]. The latter phenomenon is observed in a stochastic bistable system driven by a small periodic force. When the characteristic time scale of bistable system, the Kramers rate, coincides with the signal frequency, a considerable amplification of the signal can be observed. In other words, some kind of synchronization between the stochastic motion in the double-well potential and the external periodical force takes place.

As is well known, the theory of oscillations considers two cases of synchronization. The first is synchronization of some dynamical system by an external periodical force. If we consider a stochastic system instead of a dynamical one, this case corresponds to stochastic resonance. The second case occurs when two (or more) oscillators with different natural frequencies are coupled. It is interesting to investigate possible similar effects in coupled stochastic systems.

In the present paper we study two coupled stochastic bistable systems. Such models have been investigated before [4]. In the papers [5] globally coupled bistable oscillators have been considered. However, in the previous studies only identical bistable systems were considered. A more interesting case is when the subsystems have different parameters and, as a consequence, different characteristic time scales.

As a basic model we consider the simple overdamped bistable system which is described by the stochastic differential equation (SDE)

$$\frac{dx}{dt}=-\frac{dU(x)}{dx}+\xi(t)\ ,\quad \langle \xi(t)\xi(t+s)\rangle=2\sigma\delta(s)\ , \tag{1}$$

where $U(x)$ is the double-well potential in the form

$$U(x)=-\alpha x^2/2+x^4/4 \tag{2}$$

and σ is the intensity of white-noise source $\xi(t)$. The parameter α ($\alpha>0$) characterizes the deepness of the potential wells. The stochastic bistable system (1) has a characteristic time scale which corresponds to the mean frequency of the transition from one potential well to another. This averaged time scale is known as the Kramers rate [6]:

$$r_0=\frac{1}{\pi}[|U''(0)|U''(\alpha^{1/2})]^{1/2}\exp(-\Delta U/\sigma)\ , \tag{3}$$

where $U''(0)$ is the curvature of the well at the top of the barrier, $U''(\alpha)$ is the curvature in the bottoms of the wells, and ΔU is the barrier height.

For coupled stochastic bistable systems the SDE's have the form

$$\frac{dx}{dt}=\alpha x-x^3+\xi_1(t)+G_1(x,y)\ , \tag{4a}$$

$$\frac{dy}{dt}=\beta y-y^3+\xi_2(t)+G_2(x,y)\ , \tag{4b}$$

$$\langle \xi_i(t)\xi_j(t+s)\rangle=2\sigma\delta_{ij}\delta(s)\ . \tag{4c}$$

The last expression (4c) indicates that the noise sources $\xi_{1,2}(t)$ are statistically independent. The functions $G_{1,2}(x,y)$ define the nature of coupling. If, for instance, $G_1(x,y)=G(y)$ and $G_2(x,y)=0$ or $G_1(x,y)=0$ and $G_2(x,y)=G(x)$ then we have the case of one-directional coupling. Otherwise we will speak about mutual coupling. Note that the case of one-directional coupling may be considered as a bistable system excited by colored noise [7].

In the following we are interested in the bifurcation that changes the topology of the two-dimensional stationary probability density, the Kramers rates of the subsystems, and the cross-spectral characteristics of the process $x(t),y(t)$.

One of the characteristics suitable to investigate the synchronization phenomena is the coherence function $\gamma(\omega)$, which is defined as

$$\gamma(\omega)=\frac{|S_{xy}(\omega)|}{S_x(\omega)S_y(\omega)}\ , \tag{5}$$

where $S_{xy}(\omega)$ is the cross spectrum of the process $x(t),y(t)$. $S_x(\omega),S_y(\omega)$ are the power spectra of $x(t)$ and $y(t)$, respectively. The coherence function $\gamma(\omega)$ changes in the range $0\le\gamma(\omega)\le1$. When $\gamma(\omega)\approx1$ in some frequency region then it testifies that stochastic processes

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$x(t),y(t)$ are coherent in that frequency domain. In the present study we use numerical simulation of SDE (4) [8] and calculate the coherence function $\gamma(\omega)$.

Drawing a parallel with coupled dynamical systems, it seems to be interesting to investigate the Kramers rates in the subsystems (4a) and (4b). We calculate the Kramers rates in the subsystem numerically during simulation of the SDE: we compute the mean transition time from one potential well to another for each from the two bistable subsystems.

Let us consider first the case of mutual coupling. The Fokker-Planck equation (FPE) corresponding to the SDE's (4) has the form

$$\partial_t P(x,y,t)=\sigma[\partial_{xx}P+\partial_{yy}P] -\partial_x[(\alpha x-x^3+G_1(x,y))P] -\partial_y[(\beta y-y^3+G_2(x,y))P] \ . \qquad (6)$$

The stationary solution of the FPE (6) may be easily obtained if the coefficients of drift and diffusion satisfy the potential conditions [9]:

$$\frac{\partial G_1}{\partial y}=\frac{\partial G_2}{\partial x} \ . \qquad (7)$$

If the condition (7) is fulfilled then the stationary solution of the FPE (6) may be written in the potential form

$$P_s(x,y)=C\exp[-\Phi(x,y)] \ , \qquad (8)$$

where C is the normalization constant and the potential $\Phi(x,y)$ is

$$\Phi(x,y)=-\sigma^{-1}\int\int\{[\alpha x-x^3+G_1(x,y)]dx +[\beta y-y^3+G_2(x,y)]dy\} \ . \qquad (9)$$

The bifurcation transitions in the stochastic system (4) in this case are registered by the change of the number of extrema of the stationary probability density $P_s(x,y)$. Such changes take place in full agreement with the bifurcations of states of equilibrium of the system (4) in the absence of noise ($\sigma=0$) [2,10]. Let the functions $G_{1,2}(x,y)$ be

$$G_1(x,y)=-G_2(x,y)=D(y-x) \ . \qquad (10)$$

Such types of coupling are typical for real oscillating systems. Obviously, the functions $G_{1,2}$ defined by (11) satisfy the potential conditions. The parameter D is the coupling strength or coupling parameter. The potential $\Phi(x,y)$ is

$$\Phi(x,y)=-\sigma^{-1}[(\alpha-D)x^2/2-x^4/4+(\beta-D)y^2/2 -y^4/4+Dxy] \ . \qquad (11)$$

The states of equilibrium (x_0,y_0) of the system (4),(10) in the absence of noise are defined by the roots of equations

$$\begin{aligned}\alpha x_0-x_0^3+D(y_0-x_0)=0 \ ,\\ \beta y_0-y_0^3+D(x_0-y_0)=0 \ .\end{aligned} \qquad (12)$$

The bifurcations occurring in such systems are well known [4]. However, for further consideration we describe the bifurcation picture briefly. The bifurcation diagram of the system (4),(11) on the parameter plane (α,D) is shown in Fig. 1. The corresponding views of the two-dimensional stationary probability density $P_s(x,y)$ are shown in Figs. 2(a)–2(d) as contour lines. Region 1 on the bifurcation diagram (Fig. 1) bounded by the lines S_1 and S_2 corresponds to the existence of nine states of equilibrium. The corresponding view of $P_s(x,y)$ is shown in Fig. 2(a). There are four nodes corresponding to the maxima of $P_s(x,y)$ and four saddles corresponding to the minima of $P_s(x,y)$. In the origin there is an unstable node which corresponds to a hole in the probability density in the neighborhood of the origin. The lines S_1 and S_2 on the bifurcation diagram Fig. 1 correspond to the saddle-node bifurcation and have codimension equal to unity. The point A in Fig. 1 has codimension equal to 2 and corresponds to the merging of the states of equilibrium. This bifurcation point corresponds to the peculiarity of cusp type. The bifurcation line S_3 in Fig. 1 corresponds to the merging of state of equilibrium near the origin and is determined by the expression

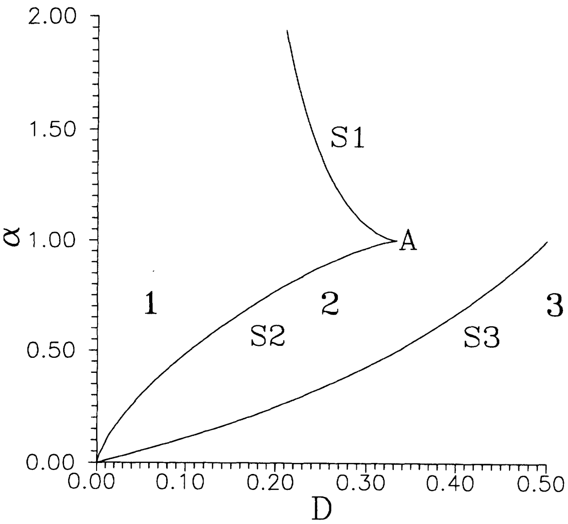

FIG. 1. Bifurcation diagram for the case of mutual coupling. The parameters β and σ are fixed: $(\beta,\sigma)=(1.0,0.1)$.

$$D=\frac{\alpha\beta}{\alpha+\beta} \ . \qquad (13)$$

While crossing this line the state of equilibrium in the origin changes its stability from the unstable node to the saddle. As a result, the hole in the origin neighborhood disappears.

Let us consider now the evolution of the coherence function $\gamma(\omega)$ with the increase of the coupling parameter D. The results of the coherence-function computations are shown in Fig. 3. When the parameter D increases, the coherence function $\gamma(\omega)$ increases too and tends to unity in the low-frequency domain. This effect demonstrates the growth of coherence degree between the processes $x(t)$ and $y(t)$. It is important to note that this effect takes place while the parameter D values are located out of region 1 on the bifurcation diagram Fig. 1. Thus crossing of the bifurcation lines S_1 and S_2 leads to the growth of coherence degree of the processes in the subsystems. In the absence of bistability in subsystems ($\alpha<0$, $\beta<0$) such an effect is not observed [see, for ex-

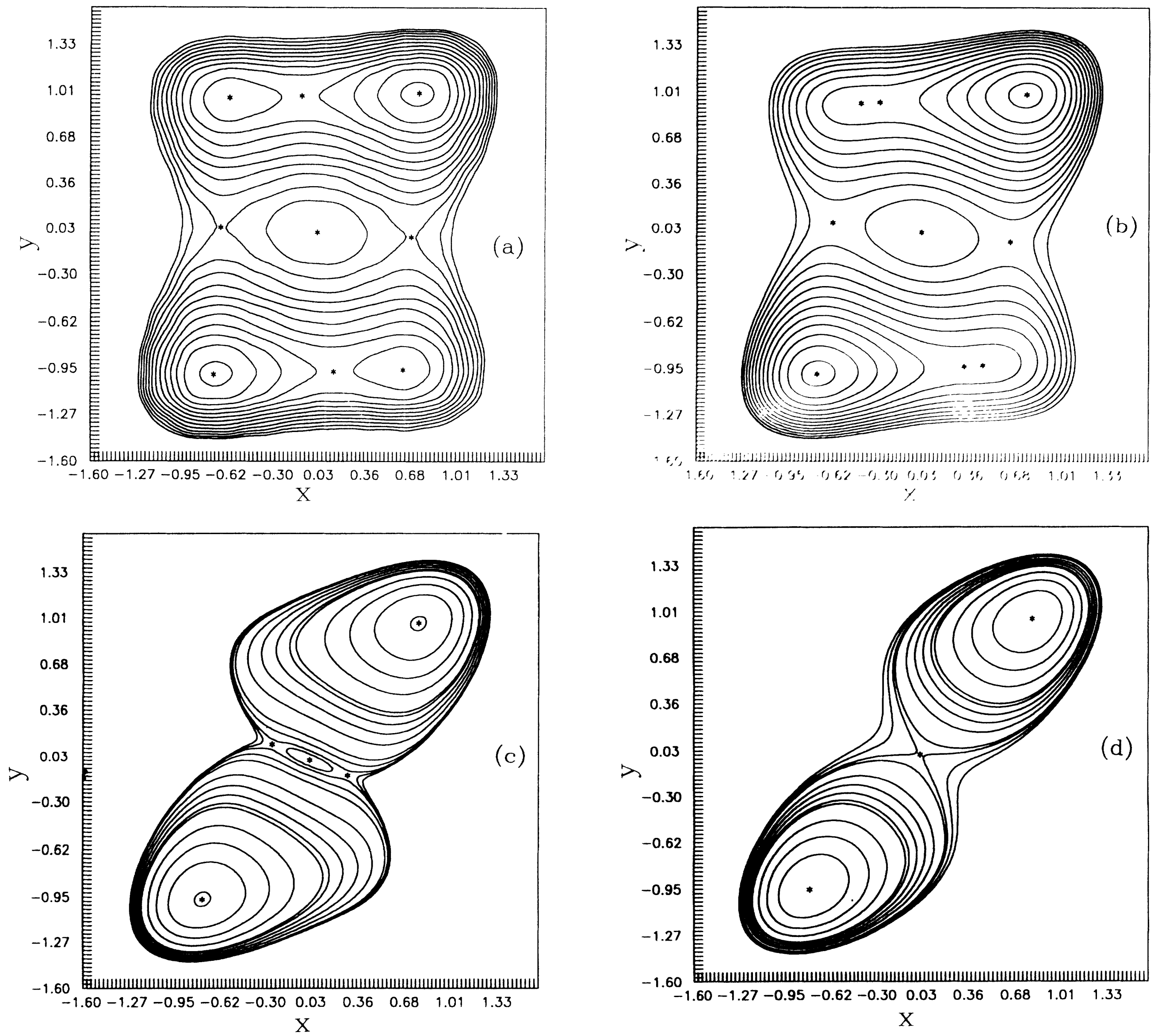

FIG. 2. Contour lines of the two-dimensional stationary probability density $P(x,y)$ for the case of mutual coupling. The parameters $(\alpha,\beta,\sigma)=(0.5,1.0,0.1)$; (a) $D=0.05$, (b) $D=0.1$, (c) $D=0.3$, (d) $D=0.4$.

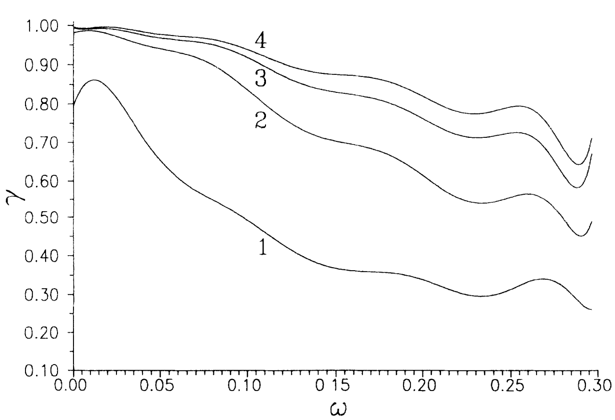

FIG. 3. Coherence function $\gamma(\omega)$ in the case of mutual coupling. The parameters are $(\alpha,\beta,\sigma)=(0.5,1.0,0.1)$; the coupling parameter D is varied: (1) $D=0.1$, (2) $D=0.3$, (3) $D=0.5$, (4) $D=0.7$.

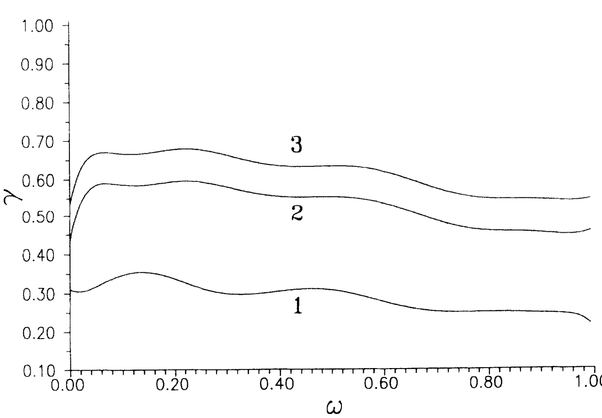

FIG. 4. Coherence function in the absence of bistability. The parameters are $(\alpha,\beta,\sigma)=(-0.2,-0.2,0.1)$; the coupling parameter D is varied: (1) $D=0.1$, (2) $D=0.7$, (3) $D=1.0$.

ample, Fig. 4, $(\alpha,\beta,\sigma)=(-0.2,-0.2,0.1)$].

The above-mentioned effect of coherence-degree growth may be easily explained. In region 1 (see Fig. 1) the stationary probability density $P_s(x,y)$ has four maxima corresponding to the wells of the potential $\Phi(x,y)$ (11). In this case transitions between these four wells are possible. As the noise sources in SDE's are statistically independent, the processes in the subsystems are practically noncoherent. The saddle-node bifurcations corresponding to the crossing of the bifurcation lines S_1 and S_2 lead to the merging of states of equilibrium and result in the existence of only two potential wells in $\Phi(x,y)$. In such a case the system becomes more symmetrical and transitions only between these two wells are possible. As a result, the processes in the subsystems become more coherent.

We consider the dependencies of the Kramers rates in subsystems r_1 and r_2 on the parameter of coupling D while the parameters $(\alpha,\beta,\sigma)=(0.5,1.0,0.1)$ are fixed. The results are presented in Fig. 5. As it is seen from the figure, Kramers rates in the subsystems draw closer to one another when the parameter D is increased. Such behavior of partial frequencies is typical for the phenomenon of synchronization of coupled classical oscillating systems in the case of frequency locking. For coupled system with noise, such effects may be called *stochastic synchronization.*

Let us consider now the case of one-directional coupling. We choose a coupling function $G(y)$ in the simplest linear form:

$$G_1(x,y)=G(y)=Dy\ , \quad G_2(x,y)=0\ . \tag{14}$$

As mentioned above, the case of the one-directional coupling may be considered as a colored-noise $y(t)$ influence on the bistable system determined by Eq. (4a). Statistical properties of the colored noise $y(t)$ are determined by Eq. (4b).

The computer simulations of SDE's (4) with the coupling function (14) showed that the results of computations of the coherence function for this case are in full agreement with results for the case of mutual coupling. As to the effect of the Kramers rates locking, then in the

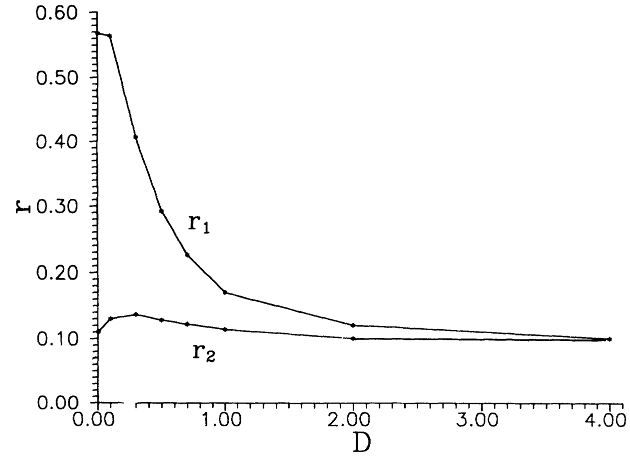

FIG. 5. Dependencies of the Kramers rates in the subsystems on the coupling parameter D; parameters $(\alpha,\beta,\sigma)=(0.5,1.0,0.1)$.

case of one-directional coupling there is no point to speak about it. In this case the increase of the coupling parameter D leads to an increase of the colored-noise $y(t)$ intensity and thus to the increase of the Kramers rate in the bistable system (4a).

The present investigations show that coupled bistable systems driven by independent noisy sources demonstrate effects similar to synchronization phenomena in classical oscillation systems. These phenomena manifest themselves in the locking Kramers rates of subsystems and in the evolution of the coherence function, reflecting growth of the coherence degree in subsystems when the coupling parameter increases. It is shown that the behavior of the coherence function is connected with the bifurcations of the extrema of the stationary probability density.

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